MODEL SOLUTIONS

Please check the examination details below before entering your candidate information

Candidate surname	Other names
MODEL SOLUTIONS	

Centre Number

Candidate Number

Pearson Edexcel Level 3 GCE

Monday 26th June 2023

Afternoon (Time: 1 hour 30 minutes)

Paper reference **9FM0/4C**

Further Mathematics

Advanced

PAPER 4C: Further Mechanics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Unless otherwise indicated, whenever a value of g is required, take $g = 9.8$ m s^{-2} and give your answer to either 2 significant figures or 3 significant figures.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

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Pearson

1. Three particles of masses $3m$, $4m$ and km are positioned at the points with coordinates $(2a, 3a)$, $(a, 5a)$ and $(2\mu a, \mu a)$ respectively, where k and μ are constants.

The centre of mass of the three particles is at the point with coordinates $(2a, 4a)$.

Find (i) the value of k

(ii) the value of μ

(6)

Take moments about the x-axis 1

$\bar{x}(\Sigma \text{ total mass}) = \Sigma(\text{distance} \times \text{mass})$

$2a(3m + 4m + km) = 2a(3m) + a(4m) + 2\mu a(km)$

$\Rightarrow \quad 2(7+k) = 10 + 2\mu k$

$\Rightarrow \quad 14 + 2k = 10 + 2\mu k$

$\Rightarrow \quad 2 = k(\mu - 1)$ 1 (1)

Take moments about the y-axis. 1

$4a(3m + 4m + km) = 3a(3m) + 5a(4m) + \mu a(km)$

$\Rightarrow \quad 28 + 4k = 29 + \mu k$

$\Rightarrow \quad 4k - \mu k = 1$

$\Rightarrow \quad k(4 - \mu) = 1$ 1 (2)

(1) ÷ (2) implies that

$\dfrac{\mu - 1}{4 - \mu} = 2 \Rightarrow \mu - 1 = 8 - 2\mu$

$\Rightarrow 3\mu = 9 \Rightarrow \mu = 3$ 1

$\Rightarrow k = 1$ 1

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2. A particle of mass 2 kg is moving in a straight line on a smooth horizontal surface under the action of a horizontal force of magnitude F newtons.

At time t seconds $(t > 0)$,

- the particle is moving with speed $v\,\text{m s}^{-1}$
- $F = 2 + v$

The time taken for the speed of the particle to increase from $5\,\text{m s}^{-1}$ to $10\,\text{m s}^{-1}$ is T seconds.

(a) Show that $T = 2\ln\dfrac{12}{7}$

(4)

The distance moved by the particle as its speed increases from $5\,\text{m s}^{-1}$ to $10\,\text{m s}^{-1}$ is D metres.

(b) Find the exact value of D.

(4)

a) Recall that $F = ma = m\dfrac{dv}{dt}$

$$2 + v = 2\frac{dv}{dt} \quad ①$$

$$\int 1\,dt = \int \frac{2}{2+v}\,dv \quad ①$$

$$\Rightarrow \quad t = 2\ln|2+v| + c \quad ①$$

when $v = 5$, $t = t_1$, when $v = 10$, $t = t_2$

$$t_1 = 2\ln(7) + c$$

$$t_2 = 2\ln(12) + c$$

$$T = t_2 - t_1 = 2\ln(12) + c - [2\ln(7) + c]$$

$$\Rightarrow \quad T = 2\ln(12) - 2\ln(7)$$

$$= 2\ln\left(\frac{12}{7}\right) \quad ①$$

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Question 2 continued

b) Recall that $F = ma = mv\frac{dv}{dx}$ by Chain Rule.

$2 + v = 2v\frac{dv}{dx}$ ①

Note that $2v = 4 + 2v - 4$

$\Rightarrow \int 1 \, dx = \int \frac{2v}{2+v} \, dv$

$\Rightarrow \int 1 \, dx = \int \frac{4+2v}{2+v} - \frac{4}{2+v} \, dv$

$\Rightarrow \int 1 \, dx = \int 2 - \frac{4}{2+v} \, dv$ ①

$\Rightarrow x = 2v - 4\ln|2+v| + c$ ①

when $v = 5$, $x = x_1$, when $v = 10$, $x = x_2$

$x_1 = 10 - 4\ln(7) + c$

$x_2 = 20 - 4\ln(12) + c$

$D = x_2 - x_1 = 20 - 4\ln(12) + c - [10 - 4\ln(7) + c]$

$= 10 - 4\ln(12) + 4\ln(7)$

$= 10 - 4\ln\left(\frac{12}{7}\right)$ ①

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3. [*In this question you may quote, without proof, the formula for the distance of the centre of mass of a uniform circular arc from its centre.*]

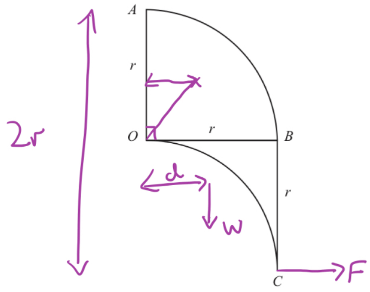

Figure 1

Five pieces of a uniform wire are joined together to form the rigid **framework** $OABCO$ shown in Figure 1, where

- OA, OB and BC are straight, with $OA = OB = BC = r$
- arc AB is one quarter of a circle with centre O and radius r
- arc OC is one quarter of a circle of radius r
- all five pieces of wire lie in the same plane

(a) Show that the centre of mass of arc AB is a distance $\frac{2r}{\pi}$ from OA.

(2)

Given that the distance of the centre of mass of the framework from OA is d,

(b) show that $d = \dfrac{7r}{2(3+\pi)}$

(4)

The framework is freely pivoted at A.

The framework is held in equilibrium, with AO vertical, by a horizontal force of magnitude F which is applied to the framework at C.

Given that the weight of the framework is W

(c) find F in terms of W

(3)

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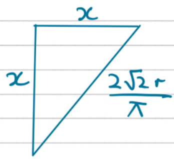
Question 3 continued

a) From the Formula Booklet, we know that a circular arc of radius r and angle at the centre 2α is $\frac{r\sin\alpha}{\alpha}$ from the centre.

$$2\alpha = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{4}$$

$$\frac{r\sin\pi/4}{\pi/4} = \frac{2\sqrt{2}r}{\pi} \quad 1$$



By Pythagoras, $2x^2 = \left(\frac{2\sqrt{2}r}{\pi}\right)^2$

$$\Rightarrow \sqrt{2}x = \frac{2\sqrt{2}r}{\pi}$$

$$\Rightarrow x = \frac{2r}{\pi} \quad 1$$

b) Take moments about OA. 1

$$d \times (\text{total mass}) = \Sigma(\text{mass} \times \text{distance})$$

$$d\left(r + r + \frac{\pi r}{2} + \frac{\pi r}{2} + r\right) = \frac{2r}{\pi}\left(\frac{\pi r}{2}\right) + \frac{2r}{\pi}\left(\frac{\pi r}{2}\right) + \frac{1}{2}r(r) + r(r)$$
1 1

Question 3 continued

$$\Rightarrow d(3r + \pi r) = \frac{7}{2} r^2$$

$$\Rightarrow d(3+\pi) = \frac{7r}{2}$$

$$\Rightarrow d = \frac{7r}{2(3+\pi)} \quad (1)$$

c) Take moments about A. (1) see diagram

$$F(2r) = \frac{7r}{2(3+\pi)} W \quad (1)$$

$$\Rightarrow F = \frac{7W}{4(3+\pi)} \quad (1)$$

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4.

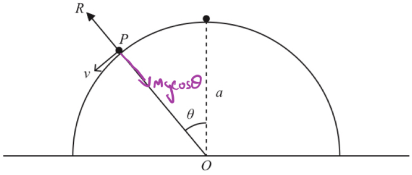

Figure 2

A smooth hemisphere of radius a is fixed on a horizontal surface with its plane face in contact with the surface. The centre of the plane face of the hemisphere is O.

A particle P of mass M is disturbed from rest at the highest point of the hemisphere.

When P is still on the surface of the hemisphere and the radius from O to P is at an angle θ to the vertical,

- the speed of P is v
- the normal reaction between the hemisphere and the particle is R, as shown in Figure 2.

(a) Show that R = Mg(3cos − 2)

(6)

(b) Find, in terms of a and g, the speed of the particle at the instant when the particle leaves the surface of the hemisphere.

(3)

a) By the conservation of energy

$Mgh_1 = Mgh_2 + \frac{1}{2}Mv^2$ as there is no initial speed.

$Mga = Mga\cos\theta + \frac{1}{2}Mv^2$ 1

$\Rightarrow 2ga(1-\cos\theta) = v^2$ (1) 1

Recall that $F = Ma \Rightarrow F = \frac{Mv^2}{r}$ as $a = \frac{v^2}{r}$ during circular motion.

$(\downarrow)$ $Mg\cos\theta - R = M\frac{v^2}{a}$ 1 1

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Question 4 continued

Sub v^2 from (1) into this

$$mg\cos\theta - R = \frac{m}{a} \cdot 2ga(1-\cos\theta) \quad ①$$

$$\Rightarrow mg\cos\theta - R = 2mg(1-\cos\theta)$$

$$\Rightarrow R = mg\cos\theta - 2mg(1-\cos\theta)$$

$$\Rightarrow R = mg\cos\theta - 2mg + 2mg\cos\theta$$

$$\Rightarrow R = mg(3\cos\theta - 2) \quad ①$$

b) The particle leaves the surface of the hemisphere when $R=0$.

$$0 = mg(3\cos\theta - 2) \quad ①$$

$$\Rightarrow mg = 0 \quad \text{or} \quad 3\cos\theta - 2 = 0$$

$$3\cos\theta - 2 = 0 \Rightarrow \cos\theta = 2/3$$

$$v^2 = 2ga(1 - 2/3) \; ① \text{ by (1)}$$

$$\Rightarrow v^2 = \frac{2}{3}ga$$

$$\Rightarrow v = \sqrt{\frac{2}{3}ag} \quad ①$$

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5.

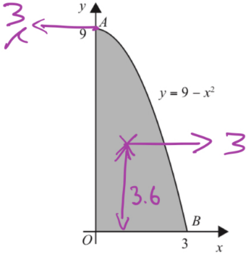

Figure 3

A uniform lamina OAB is modelled by the finite region bounded by the x-axis, the y-axis and the curve with equation $y = 9 - x^2$, for $x \geqslant 0$, as shown shaded in Figure 3. The unit of length on both axes is 1 m.

The area of the lamina is 18 m^2

(a) Show that the centre of mass of the lamina is 3.6 m from ***OB***.

[*Solutions relying on calculator technology are not acceptable.*]

(4)

A light string has one end attached to the lamina at O and the other end attached to the ceiling. A second light string has one end attached to the lamina at A and the other end attached to the ceiling.
The lamina hangs in equilibrium with the strings vertical and OA horizontal.
The weight of the lamina is W
The tension in the string attached to the lamina at A is λW

(b) Find the value of λ

(3)

a) Recall that $\bar{y} = \dfrac{\int \frac{1}{2}y^2\,dx}{\int y\,dx} = \dfrac{\int \frac{1}{2}y^2\,dx}{\text{Area}}$

$\bar{y} = \dfrac{\int_0^3 \frac{1}{2}(9-x^2)^2\,dx}{18}$ 1 1

$= \dfrac{1}{36}\int_0^3 x^4 - 18x^2 + 81\ dx$

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Question 5 continued

$$= \frac{1}{36}\left[81x - 6x^3 + \frac{x^5}{5}\right]_0^3 \quad (1)$$

$$= \frac{1}{36}\left[81(3) - 6(3)^3 + \frac{(3)^5}{5} - 0\right]$$

$$= 3.6 \quad (1)$$

b) Take moments about O (1) Use diagram.

$3.6W = 9\lambda W$ (1)

$$\Rightarrow \frac{3.6}{9} = \lambda$$

$$\Rightarrow \lambda = 0.4 \quad (1)$$

6.

$\tan\theta = \frac{0.6}{0.8} = \frac{3}{4}$

$\cos\theta = 4/5$

$\tan\theta = 3/5$

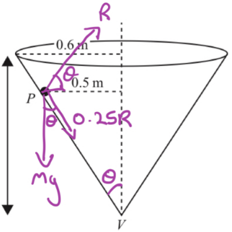

Figure 4

A hollow right circular cone, of internal base radius 0.6 m and height 0.8 m, is fixed with its axis vertical and its vertex V pointing downwards, as shown in Figure 4.

A particle P of mass m kg moves in a horizontal circle of radius 0.5 m on the rough inner surface of the cone.

The particle P moves with constant angular speed ω rad s^{-1}

The coefficient of friction between the particle P and the inner surface of the cone is 0.25

Find the greatest possible value of ω

(9)

Recall that $F_{max} = \mu R = 0.25R$

Resolve Vertically (Equilibrium) ①

$R\sin\theta = Mg + 0.25\,R\cos\theta$ ①

$\Rightarrow \frac{3}{5}R = Mg + \frac{1}{5}R$

$\Rightarrow R = \frac{5}{2}Mg$ ①

Recall that $F = Ma = Mr\omega^2$ for circular motion.

Forming an equation for motion towards the centre. ①

$0.25R\sin\theta + R\cos\theta = M\omega^2(0.5)$ ① ①

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Question 6 continued

$$\Rightarrow \left(\frac{5}{2}mg\right)\left(\frac{4}{5}\right) + \frac{1}{4}\left(\frac{5}{2}mg\right)\left(\frac{3}{5}\right) = \frac{1}{2}m\omega^2 \quad 1$$

$$\Rightarrow \frac{19}{8}g = \frac{1}{2}\omega^2 \quad 1$$

$$\Rightarrow \frac{19}{4}g = \omega^2$$

$$\Rightarrow \omega = 6.82 \quad 1$$

This is the maximum as we have used the maximum friction.

7.

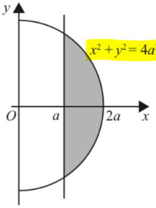
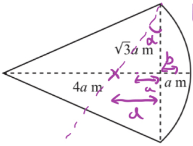


Figure 5



$b = (4\ln 2 - \frac{5}{2})a$

Figure 6

The shaded region shown in Figure 5 is bounded by the line with equation $x = a$ and the curve with equation $x^2 + y^2 = 4a^2$

This shaded region is rotated through 180° about the x-axis to form a solid of revolution.

This solid is used to model a dome with height a metres and base radius $\sqrt{3}a$ metres.

The dome is modelled as being non-uniform with the mass per unit volume of the dome at the point (x, y, z) equal to $\frac{\lambda}{x^2}$ kg m^{-3}, where $a \leqslant x \leqslant 2a$ and λ is a constant.

(a) Show that the distance of the centre of mass of the dome from the centre of its plane face is $\left(4\ln 2 - \frac{5}{2}\right)a$ metres.

(6)

A solid uniform right circular cone has base radius $\sqrt{3}a$ metres and perpendicular height $4a$ metres. A toy is formed by attaching the plane surface of the dome to the plane surface of the cone, as shown in Figure 6.

The weight of the cone is kW and the weight of the dome is $2W$

The centre of mass of the toy is a distance d metres from the plane face of the dome.

(b) Show that $d = \frac{|k + 5 - 8\ln 2|}{2 + k}a$

(4)

The toy is suspended from a point on the circumference of the plane face of the dome and hangs freely in equilibrium with the plane face of the dome at an angle α to the downward vertical.

Given that $\tan\alpha = \frac{1}{2\sqrt{3}}$

(c) find the exact value of k.

(3)

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Question 7 continued

a) Recall that

$\bar{x} = \dfrac{\pi \int x y^2 \rho \, dx}{\text{Mass}}$ because it is non-uniform and rotated around the x-axis.

and Mass $= \pi \int y^2 \rho \, dx$

$M = \pi \int y^2 \rho \, dx$

$\Rightarrow M = \pi \int_a^{2a} (4a^2 - x^2)\left(\dfrac{\lambda}{x^2}\right) dx$ 1

$= \pi\lambda \int_a^{2a} 4a^2 x^{-2} - 1 \, dx$

$= \pi\lambda \left[-4a^2 x^{-1} - x\right]_a^{2a}$

$= \pi\lambda \left[-4a^2 (2a)^{-1} - (2a) + 4a^2 (a)^{-1} + a\right]$

$= \pi\lambda a$ 1

$\Rightarrow \pi\lambda a \bar{x} = \pi \int_a^{2a} (4a^2 - x^2)\left(\dfrac{\lambda}{x^2}\right) x \, dx$ 1

$\Rightarrow a\bar{x} = \int_a^{2a} 4a^2 x^{-1} - x \, dx$

$\Rightarrow a\bar{x} = \left[4a^2 \ln|x| - \dfrac{x^2}{2}\right]_a^{2a}$

Question 7 continued

$\Rightarrow a\bar{x} = 4a^2\ln|2a| - 2a^2 - 4a^2\ln|a| \quad \frac{a^2}{2}$ ①

$\Rightarrow a\bar{x} = 4a^2\ln 2 - \frac{3a^2}{2}$

$\Rightarrow \bar{x} = 4a\ln 2 - \frac{3a}{2}$ ①

This is from the origin, so from the centre of its plane face, we have

$\bar{x}' = 4a\ln 2 - \frac{3a}{2} - a$

$\Rightarrow \bar{x}' = (4\ln 2 - 5/2)a$ ①

b) Using the Formula Booklet, we have that the centre of mass of a solid cone is $1/4 h$ above the base.

Here, $1/4\ h$ is $\frac{4a}{4} = a$. ①

We take moments about a diameter of the plane face. ①

$d(\text{total mass}) = \Sigma(\text{mass} \times \text{distance})$

$d(2w + kw) = a(kw) - a(4\ln 2 - 5/2)(2w)$ ①

$\Rightarrow d(2+k) = ak - a(8\ln 2 - 5)$

$\Rightarrow d = \frac{|k + 5 - 8\ln 2|}{2+k}a$ ①

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Question 7 continued

c) See diagram

$$\tan\alpha = \frac{\frac{a(k-8\ln 2+5)}{2+k}}{\sqrt{3}a} = \frac{1}{2\sqrt{3}} \quad 1 \quad 1$$

$$\Rightarrow \frac{k-8\ln 2+5}{2+k} = \frac{1}{2}$$

$$\Rightarrow 2k-16\ln 2+10 = 2+k$$

$$\Rightarrow k = 16\ln 2 - 8 \quad 1$$

(Total for Question 7 is 13 marks)

8.

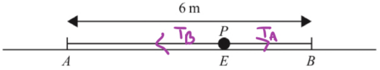

Figure 7

The fixed points A and B lie on a smooth horizontal surface with $AB = 6$ m.

A particle P has mass 0.3 kg.

One end of a light elastic string, of natural length 2 m and modulus of elasticity 20 N, is attached to P, and the other end is attached to A.

One end of another light elastic string, of natural length 2 m and modulus of elasticity 40 N, is attached to P and the other end is attached to B.

The particle P is at rest in equilibrium at the point E on the surface, as shown in Figure 7.

(a) Show that $EB = \frac{8}{3}$ m.

(3)

The particle P is now held at the midpoint of AB and released from rest.

(b) Show that P oscillates with simple harmonic motion about the point E.

(4)

The time between the instant when P is released and the instant when it first returns to the point E is S seconds.

(c) Find the exact value of S.

(3)

(d) Find the length of time during one oscillation for which the speed of P is more than $2\,\text{m s}^{-1}$

(4)

a) Recall that $T = \frac{\lambda e}{l}$

In equilibrium, $T_A = T_B$

$\frac{40e}{2} = \frac{20(2-e)}{2}$

The extension of T_B is 6 minus however far P is to the right, which is th extension of T_A.

$\Rightarrow \quad 40e = 40 - 20e$

$\Rightarrow \quad 60e = 40 \quad \Rightarrow e = 2/3$

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Question 8 continued

So $EB = 6 + 2/3 = 8/3$.

b) Recall that $F = ma = m\ddot{x}$

$T_A - T_B = m\ddot{x}$

$\Rightarrow \dfrac{40\left(2/3 + x\right)}{2} - \dfrac{20\left(4/3 - x\right)}{2} = -0.3\ddot{x}$

$\Rightarrow \dfrac{80}{3} + 20x - \dfrac{80}{3} + 10x = -0.3\ddot{x}$ acceleration acts in the opposite way

$\Rightarrow 30x = -0.3\ddot{x}$

$\Rightarrow \ddot{x} = -100x$

which is of the form $\ddot{x} = -w^2 x$ hence SHM

c) Recall that Period $= \dfrac{2\pi}{\omega}$

Period $= \dfrac{2\pi}{10} = \dfrac{\pi}{5}$

First reaches E at a quater of a period.

$S = \dfrac{1}{4} \times \dfrac{\pi}{5} = \dfrac{\pi}{20}$

P 7 4 0 8 5 A 0 2 9 3 2

Question 8 continued

d) Recall that $v^2 = w^2(A^2 - x^2)$ and $x = A\cos(wt)$ at the end points.

$A = 1/3$ (Midpoint to E)

$4 = 100(1/9 - x^2)$ (1)

$\Rightarrow x^2 = \frac{16}{225} \Rightarrow x = \pm\frac{4}{15}$ (1)

$\frac{4}{15} = \frac{1}{3}\cos 10t_1 \Rightarrow t_1 = 0.06435$

$-\frac{4}{15} = \frac{1}{3}\cos 10t_2 \Rightarrow t_2 = 0.2498$

$t_2 - t_1 = 0.18545$ (1)

This is the amount of time where the speed of p is more than $2\,ms^{-1}$ in half an oscillation.

So $t = 2(0.18545) = 0.371$ (1)